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# INELASTIC BEHAVIOR OF REINFORCED CONCRETE MEMBERS SUBJECTED TO SHORT-TIME STATIC LOADS

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ENGINEERING MECHANICS DIVISION
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# INELASTIC BEHAVIOR OF REINFORCED CONCRETE MEMBERS SUBJECTED TO SHORT-TIME STATIC LOADS

L. H. N. Lee,[1] Jr. M. ASCE

## Synopsis

An experimentally determined stress-strain relationship for concrete in compression due to flexure is presented and developed. A theory of inelastic flexure based on this is formulated to explain the behavior of reinforced concrete members and to predict ultimate loads. Close agreement is found between the results of the theory and the experimental data reported in the literature. A method of determining the collapse loads of statically indeterminate beams is given and is used in checking the test results obtained for three statically indeterminate beams.

## Introduction

In the analysis of reinforced concrete members, one of the factors which is difficult to deal with is plastic flow. The stress-strain relation of concrete is affected by time, previous history of stress and strain, and the restraining conditions. This uncertainty is one of the major causes which inclines the modern inelastic theories towards semi-empirical forms.

In order that a continuous frame may fail simultaneously at many places, its design must be based not only on the ultimate strengths of individual sections, but also on the moment-deformation relations along the axis of the frame. The theory of elasticity gives a good picture of the distribution of moments in the structure under design load; but, because of the non-linear characteristics of the material, this theory cannot be used to determine the collapse load of the structure. Hence, the factor of safety based on ultimate loads is uncertain.

To derive the moment-deformation relation, the stress-strain relation should be established first. The writer has formulated a general stress-strain relation on the basis of various tests of compression cylinders and of beams under short-time static load, for which conditions the uncertainty due to plastic flow is eliminated.

The letter symbors used in the paper are defined when they are first introduced.

## Stress-Strain Relation for Concrete in Compression

It is universally recognized that the typical stress-strain curve for concrete tested in a short-time, static, standard compression test, up to maximum stress, is a portion of a parabola oriented with axis vertical and vertex upward. The standard cylinder test does not determine the shape of the stress-strain curve beyond the point of maximum stress. However, research to

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determine the shape of the curve beyond the maximum stress has been reported during the past decade.[2,3,4,5] The results of tests reported by the Bureau of Reclamation[6,7] show that the curve appears to be close to a second degree parabolic curve up to a strain of about one and one-half times the strain corresponding to the maximum stress.

The stress-strain relation for concrete in compression due to flexure has been discussed considerably in the past. In recent years, methods[8,9,10] have been developed to determine the complete stress-strain relation directly in the beam tests. By means of a small pressure cell, direct measurements of the stresses occuring in a concrete beam were obtained by the Bureau of Reclamation.[11] The results showed that although the strains were linear up to failure, the distribution of compressive stresses were curvilinear. Another significant fact revealed was that the measured ultimate stress was equal to the cylinder strength. Similar results were found by use of the other methods and it has been concluded that the stress-strain relation for concrete in compression due to flexure is essentially similar to that obtained in direct compression.[12] In order to confirm this matter further, tests on three beams

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2. "Der Beiwert n", by F. V. Emperger, Beton and Eisen, Vol. 35, No. 19, Oct. 1936, pp. 324-32.
3. "Bruchzustand und Sicherbeit im Eisenbetonbalken", by R. Saliger, Beton und Eisen, Vol. 35, Nos. 19 and 20, Oct. 1936, pp. 317-20, and 339-46.
4. "A Comparison of Physical Properties of Concrete Made of Three Varieties of Coarse Aggregate," a thesis by Oscar E. Kiendly and Joe A. Maldari, presented to the University of Wisconsin, Madison, Wisconsin, in 1938, in Partial fulfillment of the requirement for the degree of Bachelor of Science in Civil Eng.
5. "A Modified Plastic Theory of Reinforced Concrete", by Paul Anderson and Hwa-Ni Lee, University of Minnesota Engineering Experiment Station Bulletin, No. 33, 1951, 40 pp.
6. "Stress-Strain Curves for Concrete Strained Beyond the Ultimate Load", by D. Ramalay and D. McHenry, Lab. Report No. SP-12, U.S. Bureau of Reclamation, March 1947, 22 pp.
7. "Plastic Flow of Concrete Relieves High Load Stress Concentrations", by R. F. Blanks and D. McHenry, Civil Engineering Vol. 19, No. 5, May 1949, pp. 320-22.
8. "Studies of Compressive stress Distribution in Simply Reinforced Concrete near the point of Failure", by Lester A. Herr and L. E. Vandergrift, Ohio State Univ. Eng. Exp. Station Bulletin No. 144, 1951, 15 pp.
9. "The Distribution of Concrete Stress in Reinforced and Prestressed Concrete Beams When Tested to Destruction by a Pure Bending Moment", by J. M. Prentis, Magazine of Concrete Research, No. 5, Jan. 1951, London, pp. 73-77.
10. "Recent Research in Reinforced Concrete, and Its Application to Design", by A. L. C. Baker, Journal of Inst. C. E., Vol. 35-36, 1951, pp. 262-329.
11. "Discussion on "Review of Research on Ultimate Strength of Reinforced Concrete Members", by C. P. Siess", by A. L. Parme, ACI Journal, Vol. 23, No. 10, June 1952, pp. 862-864.
12. "Inelastic Deformation of Reinforced Concrete in Relation to Ultimate Strength", by H. J. Cowan, Engineering, Vol. 174, No. 4518, London, August 1952, pp. 276-278.

were carried out by the writer, and a method similar to the methods by Nadai[13] and Prentis[9] was used to analyze the test results.

The derivation of the method is based on the following assumptions.

(1) It is assumed that the plane sections of the members remain plane throughout the entire range of flexure. In certain restricted regions these sections undoubtedly become warped by shear, but on the whole the assumption is confirmed by many investigators to be substantially true.

(2) It is assumed that there is no creep of concrete during the test period. This assumption is considered true only if the entire test is carried out continuously within approximately twenty minutes, and the duration of the test at high loads, from 60 to 100 percent of the ultimate load, is limited to five minutes. It has been found that the short-time creep of concrete in a cylinder test is small with lower loads, but the creep is large with higher loads;[14] however, due to the restraining effect of the concrete close to the neutral axis the creep of the concrete in a beam test is less.

(3) It is assumed that the tensile stress in concrete is negligible. This assumption is not valid if the value of extreme fiber tensile strain is lower than the value of the ultimate tensile strain of concrete. Once the value of tensile strain reaches the value of ultimate tensile strain of concrete which varies from $1 \times 10^{-4}$ to $2 \times 10^{-4}$ fine cracks are formed.[12] Because of the brittle characteristics of concrete in tension and the stress concentrations the cracks will propagate towards the neutral axis; once the cracks approach the neutral axis, the assumption is good.

Consider a rectangular beam reinforced with intermediate grade structural steel in tension as shown in Fig. 1. The general stress-strain relation of concrete may be expressed as

$$f = f(\epsilon) \tag{1}$$

The condition of horizontal forces in equilibrium requires that

$$\frac{kbd}{\epsilon_c}\int_0^{\epsilon_c} f(\epsilon)\, d\epsilon = A_s E_s \epsilon_s \quad \text{for} \quad 0 < \epsilon_s < \epsilon_y \tag{2}$$

Differentiating both sides of Eq. (2) with respect to $\epsilon_c$ and rearranging, one obtains

$$f_c = pE_s \left(\epsilon_c \frac{d\epsilon_s}{d\epsilon_c} + 2\epsilon_s \frac{d\epsilon_s}{d\epsilon_c} + \epsilon_s\right), \quad \text{for} \quad 0 < \epsilon_s < \epsilon_y \tag{3}$$

in which,

$$p = \frac{A_s}{bd}$$

13. "Theory of Flow and Fracture of Solids," by A. Nadai, McGraw-Hill Co., New York, 1950, Sec. 22-2, p. 356.
14. "Short-time Creep Tests of Concrete in Compression," by R. S. Jensen and F. E. Richart, Proceedings ASTM, Vol. 38, 1938, pp. 410-17.
12. Ibid.

If the steel strain has reached the yield point strain $\epsilon_y$ the extreme fiber stress $f_c$ of concrete can be found similarly by the equation

$$f_c = pf_y\left[1 + \frac{d\epsilon_s}{d\epsilon_c}\right] \quad \text{for} \quad \epsilon_y < \epsilon_s < \epsilon_{wh} \tag{4}$$

where $f_y$ is the yield point stress of steel.

These relations are not restricted to a section subjected to pure bending only; the method can be used at any section if the condition of the plane section remains plane is justified. In Fig. 2 curve A is the average stress-strain curve obtained in the four standard cylinder tests, and curve B is the average stress-strain curve derived from the three beam tests by utilizing the Eqs. (3) and (4). The beams and cylinders were made of the same concrete mix. The tests are described in the Appendix and the sample calculations are given in Table 1, Appendix. As shown in Fig. 2, the average maximum stress, 3810 psi from the beam tests, is very close to the average cylinder strength 3800 psi. There is a discrepancy between the strains corresponding to these stresses, $2240 \times 10^{-6}$ and $1920 \times 10^{-6}$ respectively, because the duration of the beam test was greater than that of the cylinder test. The stress-strain relation beyond the point of maximum stress was not determined, because the strains changed swiftly at that stage, and it was difficult to get any significant measurements without the aid of automatic devices. From the facts revealed, it is believed that the stress-strain relation for concrete in compression due to flexure is essentially similar to that obtained in direct compression.

## DEVELOPMENT OF THEORY

### Basic Assumptions

Additional assumptions to those already stated are as follows:

1. The stress-strain relation for concrete in compression due to flexure is assumed to be expressed by the equation:

$$f_c = A\epsilon - B\epsilon^2 \qquad \text{for} \qquad 0 < \epsilon < 2\epsilon_o \tag{5}$$

with

$$A = \frac{2f_c'}{\epsilon_o} \quad \text{and} \quad B = \frac{f_c'}{\epsilon_o^2} \tag{5a}$$

In Equation 5(a), $f_c'$ is the maximum stress and is the corresponding strain. (See Fig. 3).

2. It is assumed that the concrete is homogeneous and isotropic.
3. It is assumed that the members are so designed as to prevent all failures except flexural failure.
4. It is assumed that the stress-strain relation for the intermediate grade steel is of the form shown in Fig. 4.

Based on the above assumptions the following formulas may be developed with the aid of equilibrium and compatibility equations.

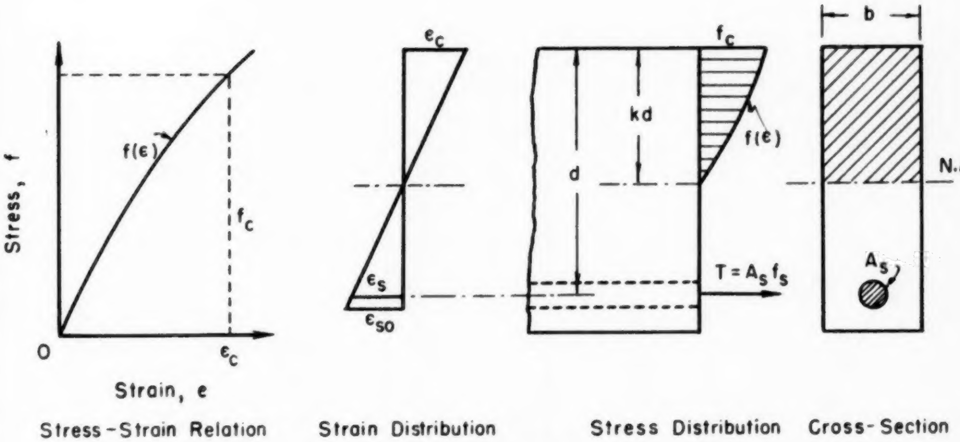


Fig. 1. Stress-Strain Relation of Concrete in Bending.

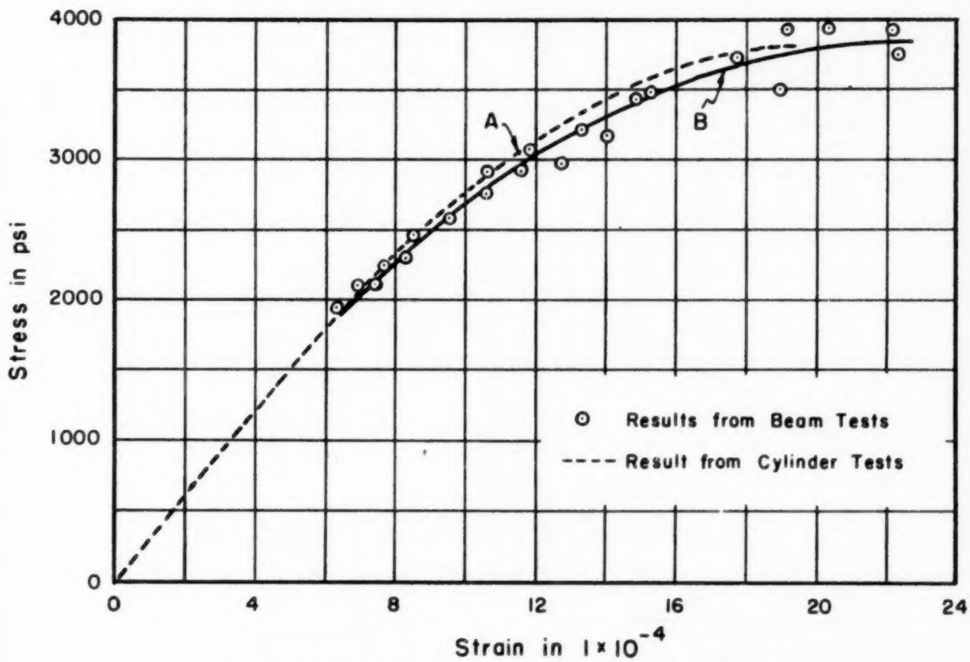


Fig. 2. Stress-Strain Relation from Bending Tests.

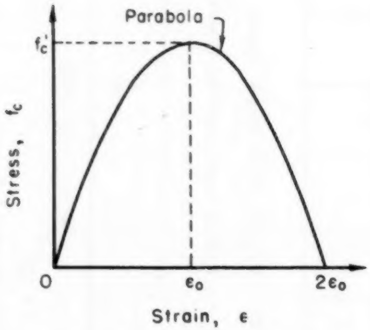


Fig. 3. Idealized Stress-Strain Diagram of Concrete Subjected to Short-Time Static Load.

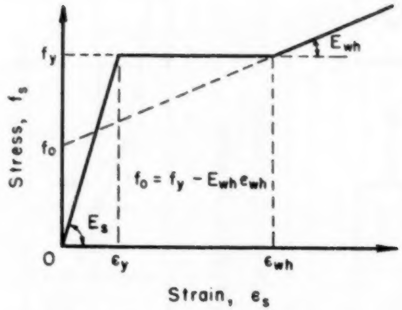


Fig. 4. Idealized Stress-Strain Diagram of Steel.

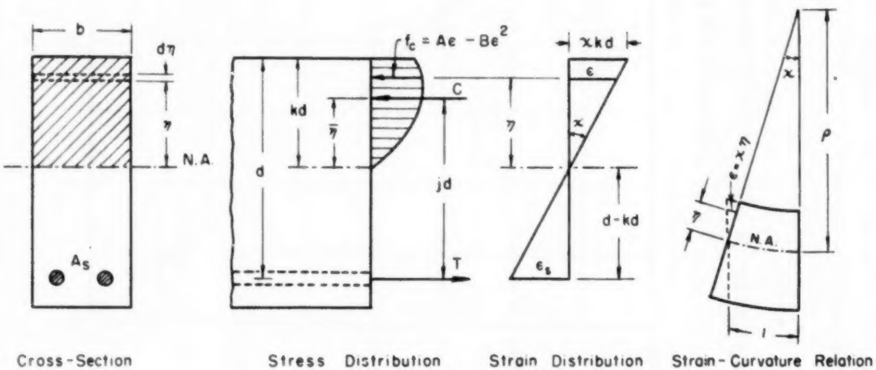


Fig. 5. Rectangular Section in Bending.

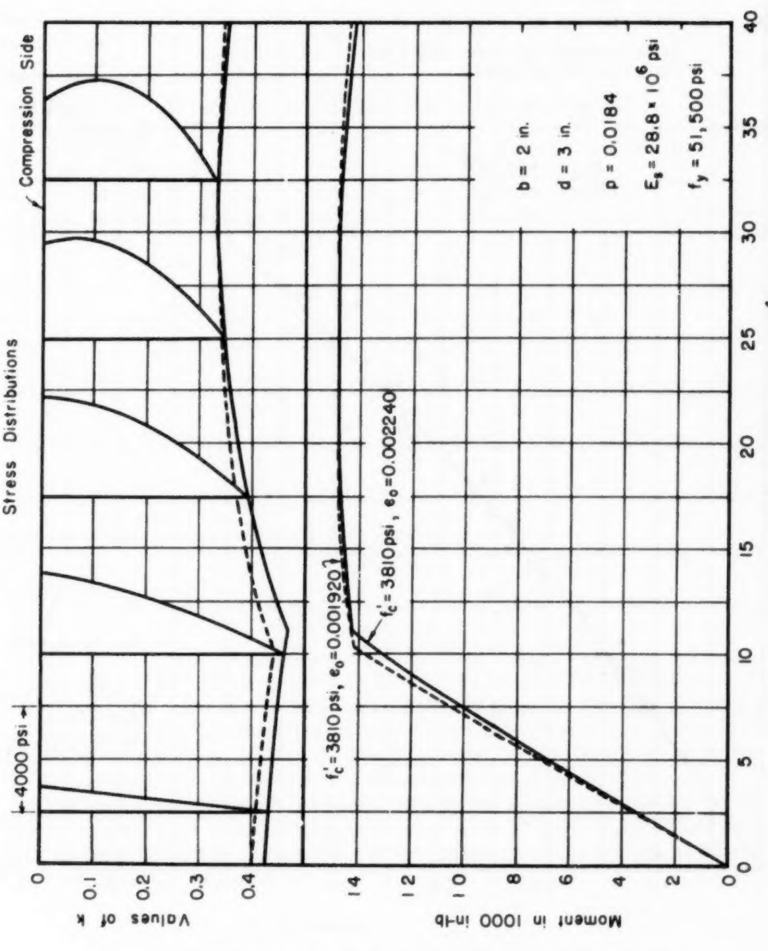


**Fig. 6. Moment-Curvature Relations and Stress Distributions.**

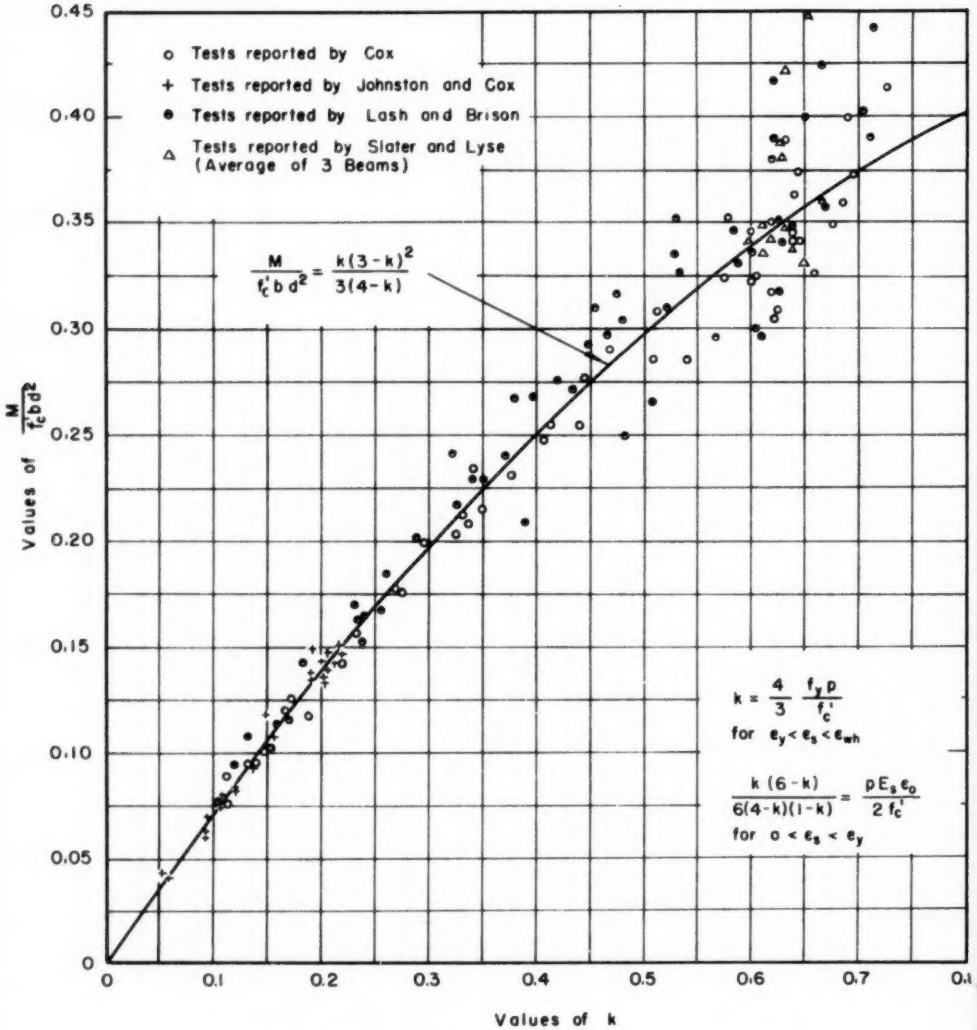


**Fig. 7. Theoretical Ultimate Moment of Concrete Beams Compared with Test Results.**

### Moment-Curvature Relations of Rectangular Beams Reinforced in Tension Only

On the assumption that plane sections remain plane, it is found that the strain $\epsilon$ at a distance $\eta$ from the neutral axis of a rectangular beam as shown in Fig. 5 is

$$\epsilon = \frac{\eta}{\rho} = \chi \eta \tag{6}$$

in which $\chi$ is the curvature. Thus Eq. (5) can be written as:

$$f_c = A\chi\eta - B\chi^2\eta^2 \tag{7}$$

By use of Eq. (7), and by the condition of horizontal equilibrium, it is found that

$$k^2\left(\frac{A}{2} - \frac{B\chi kd}{3}\right) = pE_s(1 - k) \qquad \text{for} \qquad 0 < \epsilon_s < \epsilon_y \tag{8a}$$

$$\chi k^2 d\left(\frac{A}{2} - \frac{B\chi kd}{3}\right) = pf_y \qquad \text{for} \qquad \epsilon_y < \epsilon_s < \epsilon_{wh} \tag{8b}$$

$$\chi k^2 d\left(\frac{A}{2} - \frac{B\chi kd}{3}\right) = p\left\{f_o + E_{wh}\, d(1 - k)\chi\right\} \qquad \text{for} \qquad \epsilon_s > \epsilon_{wh} \tag{8c}$$

(8)

The resisting moment of the beam section is

$$M = b\, d^3 k^2 \chi\left\{\frac{A}{6}\,(3 - k) - \frac{B}{12}\chi kd\,(4 - k)\right\} \tag{9}$$

Eq. (9) shows that the resisting moment of the section is a function of two variables, the curvature $\chi$ and the ratio k. In case the bending moment at the section is known, the values of $\chi$ and k associated with this moment can be found by solving Eqs. (8) and (9) simultaneously. The stress distribution due to this moment can be determined by substituting $\chi$ in Eq. (7).

The experimental method of determining the moment-curvature relation is quite simple. The bending moment at any section of a simply supported beam can be calculated from the loads. The corresponding curvature at the section may be found by the equation

$$\chi = \frac{\epsilon_c + \epsilon_s}{d} \tag{10}$$

in which $\epsilon_c$ and $\epsilon_s$ are the measured strains from the gages at the surface of the tensile steel and at the compression side of the concrete respectively. The quantity d is the distance between these two gages. It is not simple to determine the stress distribution experimentally. The method previously presented can be used to determine the stress distribution, if the concrete is assumed to be homogeneous and isotropic.

Theoretical moment-curvature relations and stress distributions for a rectangular beam tested by the writer are given in Fig. 6. The curves were obtained by substituting the respective values from the data of the beam in the Eqs. (7), (8) and (9). For comparison, two sets of curves are plotted in Fig. 6. One set, with value of $\epsilon_0 = 2240 \times 10^{-6}$ obtained in flexure tests, is shown in solid lines; and another set, with value of $\epsilon_0 = 1920 \times 10^{-6}$ obtained in cylinder tests, is shown in dash lines. It shows that the change of the value of $\epsilon_0$ changes the stress distributions and positions of neutral axis, but does not change the moment-curvature relation much. These theoretical results conform closely to the actual behavior of the beam. It is also interesting to note that the ultimate resisting moment of the beam in the present analysis is 14,700 in-lb which agrees precisely with the values computed by Whitney's "Plastic Theory,"[15] and Talbot's, "Parabolic Formula"[16].

## Ultimate Resisting Moments of Rectangular Beams

Denoting the relationship between M, $\chi$ and k, Eqs. (8) and (9) may be written as follows:

$$F(\chi, k) = 0 \tag{8d}$$

$$M = M(\chi, k) \tag{9a}$$

In order to determine a certain pair of values, $\chi$ and k, which will satisfy Eq. (8) and give the maximum resisting moment, it is necessary to solve the following two equations simultaneously:

$$F(\chi, k) = 0 \tag{8d}$$

$$\frac{\partial M}{\partial \chi} + \frac{\partial M}{\partial k}\frac{dk}{d\chi} = 0 \tag{9b}$$

in which

$$\frac{dk}{d\chi} = -\frac{\dfrac{\partial F}{\partial \chi}}{\dfrac{\partial F}{\partial k}} \tag{8e}$$

Exact solution of these equations is tedious; however, using the fact that k is a slowly varying function of $\chi$ (see Fig. 6), one obtains a good approximate solution by setting $\frac{dk}{d\chi} = 0$, then

$$\frac{\partial M}{\partial \chi} = 0; \qquad \chi = \frac{2\epsilon_0(3 - k)}{kd\,(4 - k)} \tag{11}$$

Substituting this value for $\chi$ in Equation (9) gives the ultimate resisting moment

15. "Plastic Theory in Reinforced Concrete Design", by C. S. Whitney, Transactions ASCE, Vol. 107, 1942, pp. 251-326.
16. "Tests of Reinforced Concrete Beams", by A. N. Talbot, University of Illinois Engineering Experiment Station Bulletin No. 1, Sept. 1904, 64 pp; Bulletin No. 4, April 1906, 84 pp.

$$M = \frac{k(3 - k)^2}{3(4 - k)} \quad f_c' \, bd^2 \tag{12}$$

The corresponding value of k which indicates the position of the neutral axis at the ultimate stage may be obtained by eliminating $\chi$ between Eqs. (8) and (11). After simplifications, it is found that

$$\frac{k^2(6 - k)}{6(4 - k)(1 - k)} = \frac{pE_s \epsilon_o}{2f_c'} \quad \text{for} \quad 0 < \epsilon_s < \epsilon_y \tag{13a}$$

$$k = \frac{4}{3} \frac{f_y p}{f_c'} \quad \text{for} \quad \epsilon_y < \epsilon_s < \epsilon_{wh} \tag{13b}$$

(13)

$$k^2 - \left(\frac{4}{3} \frac{f_o}{f_c'} - \frac{2E_{wh}\epsilon_o}{f_c'}\right) pk$$

$$- \frac{2E_{wh}\epsilon_o}{f_c'} \quad p = 0 \quad \text{for} \quad \epsilon_s > \epsilon_{wh} \tag{13c}$$

The critical steel percentage defined conventionally as the percentage for which the yield point stress of steel will be reached at the ultimate resisting moment, is determined by

$$p_o = \frac{3}{4} \frac{f_c'}{f_y} \left[ 2 - \sqrt{\frac{4\epsilon_y + 2\epsilon_o}{\epsilon_y + \epsilon_o}} \right] \tag{14}$$

A comparison of the resutls of 218 beam tests with the ultimate resisting moments computed by the foregoing formulas is shown in Fig. 7. No attempt was made to compare all the beam tests, only four well controlled series were included.[17,18,19,20] In the reports concerning these four series of tests, the values of cylinder strength $f_c'$ are given, but the values of corresponding strain are not given. In the present analysis, the values of $\epsilon_o$ are required, therefore, it is assumed that

$$\epsilon_o = 1220 + \frac{1}{6} f_c' \tag{15}$$

in which, $f_c'$ is in psi and $\epsilon_o$ is in micro-inch per inch.

17. "Compressive Strength of Concrete in Flexure", by W. A. Slater and I. Lyse, ACI Journal, Proceedings Vol. 26, June 1930; pp. 831-74.
18. "Tests of Reinforced Concrete Beams with Recommendations for Attaining Balanced Design", by K. C. Cox, ACI Journal, Sept. 1941; Proceedings Vol. 38, pp. 65-80.
19. "The Ultimate Strength of Reinforced Concrete Beams", by S.D. Lash and F. W. Brison, ACI Journal, Feb. 1950; pp. 457-72, Proceedings Vol. 46.
20. "High Yield Point Steel as Tension Reinforcement in Beams", Proceedings Vol. 36, pp. 65-80.

The tests by Johnston and Cox[20] were made on 32 beams reinforced with low steel percentages. In this series both ultimate and yield point loads were recorded. It is the writer's belief that the steel was stretched into the work hardening range at the ultimate loads. Because there is insufficient information concerning the properties of steel in the work hardening range, the recorded yield point moments were compared with the moments computed by Eqs. (12) and (13).

The average ratio of recorded to predicted ultimate resisting moments for the four series of beams was 1.00, the standard deviation of the ratio being $\pm .062$.

## ECCENTRICALLY LOAD COLUMNS

### Load-Curvature Relation of Rectangular Sections

A rectangular section with reinforcement at opposite faces loaded in the plane of symmetry is shown in Fig. 8. The total compression in the concrete of this section may be expressed as

$$C = b\chi k^2t^2\left(\frac{A}{2} - \frac{B\chi kt}{3}\right) \qquad \text{for} \qquad 0 < k < 1 \tag{16}$$

Projecting all the forces in Fig. 8 on a vertical line gives

$$P = b\chi k^2t^2\left(\frac{A}{2} - \frac{B\chi kt}{3}\right) + A_s'(f_s' - f_{cs}) - A_sf_s \tag{17}$$

An examination of Fig. 8 shows that the strains are given by

$$\epsilon_s' = \chi(kt - d'') \tag{18a}$$

$$\epsilon_s = \chi(d - kt) \tag{18b}$$

The corresponding stresses can be found from the stress-strain relations

$$\begin{aligned}
& f_s' = E_s'\,\epsilon_s' && \text{for } \epsilon_s' < \epsilon_y' \\
& f_s' = f_y' && \text{for } \epsilon_s' > \epsilon_y' \\
\text{and}\quad & f_s = E_s\epsilon_s && \text{for } \epsilon_s < \epsilon_y \\
& f_s = f_y && \text{for } \epsilon_s > \epsilon_y \\
\text{and}\quad & f_{cs} = A\epsilon_s' - B\epsilon_s'^2
\end{aligned} \tag{18c}$$

in which $f_{cs}$ is the equivalent compressive stress in concrete at the position of compression reinforcement. Taking moments of all forces about an axis at the line of action of external load gives

$$\chi k^2t\left(\frac{A}{2} - \frac{B\chi kt}{3}\right)\left(k\,\frac{4A - 3B\chi kt}{6A - 4B\chi kt} + \frac{1}{2} - k - \frac{e}{t}\right) + pf_s\left(\frac{1}{2} - \frac{d'}{t} + \frac{e}{t}\right) + p'(f_s' - f_{cs})\left(\frac{1}{2} - \frac{d''}{t} - \frac{e}{t}\right) = 0 \tag{19}$$

in which

$$p = \frac{A_s}{bt}, \qquad p' = \frac{A_s'}{bt}$$

Equations (17) (18) and (19) give a theoretical relation between the external load P, the position of the neutral axis k, and the curvature $\chi$.

Equations for Ultimate Loads of Rectangular Sections

The ultimate load for this case can be closely approximated if Eq. (16) is defferentiated with respect to $\chi$, the derivative is placed equal to zero, and $\chi$ is solved for as follows:

$$\frac{\partial C}{\partial \chi} = 0, \qquad \chi = \frac{1.5}{kt}\epsilon_0 \quad \text{for} \quad 0 < k < 1 \tag{20}$$

Substituting Eq. (20) in Eq. (17) gives the ultimate load

$$P = \frac{3}{4} f_c' \, kbt + A_s'(f_s' - f_{cs}) - A_s f_s \tag{21}$$

The corresponding value of k can be obtained by solving the equation

$$k^2 + \frac{12}{5}\left(\frac{e}{t} - \frac{1}{2}\right)k + \frac{16}{5f_c'}\left[p'(f_s' - f_{cs})\left(\frac{e}{t} - \frac{1}{2} + \frac{d''}{t}\right) - pf_s\left(\frac{e}{t} + \frac{1}{2} - \frac{d'}{t}\right)\right] = 0 \tag{22}$$

The respective stresses can be determined by finding the corresponding strains

$$\epsilon_s' = 1.5\,\epsilon_0\left(1 - \frac{d''}{kt}\right)$$

and

$$\epsilon_s = 1.5\,\epsilon_0\left(\frac{d}{kt} - 1\right) \tag{23}$$

The total eccentricity e in the foregoing formulas is the sum of the original eccentricity and the maximum deflection $\delta$ of a column. The value of $\delta$ may be found by knowing the moment-curvature relation of the column and the restraining conditions at the ends. In case the characteristics of the deflection curve are known, the value of $\delta$ may be obtained by an approximate method. For example, the deflection curve of a column hinged at both ends is very close to a sine wave, which was shown experimentally by Hognestad.[21] The curvature of the sine wave at mid-height of the column of length $l$ is

$$\frac{d^2y}{dx^2} = \frac{\delta\pi^2}{l^2} \tag{24}$$

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21. "A Study of Combined Bending and Axial Load in Reinforced Concrete Members", by E. Hognestad, University of Illinois Eng. Experiment Station Bulletin No. 339, Nov. 1951, 128 pp.

Equating Eqs. (20) and (24) gives the maximum deflection of a rectangular column

$$\delta = \frac{1.5\epsilon_0 \ell^2}{\pi^2 kt} \tag{25}$$

for the case of the neutral axis within the section at the ultimate load.

## Equations for Ultimate Loads of Cylindrical Spiral Columns

The moment-curvature relations and the equations for the ultimate loads of eccentrically loaded cylindrical spiral columns may be derived on the basis of the general assumptions presented. With reference to Fig. 9 the following equations for ultimate loading are given. Ultimate load:

$$P = C_c(k) f_c' D^2 + \sum A_s'(f_s' - f_{cs}) - \sum f_s A_s \tag{26}$$

with

$$C_c(k) f_c' D^2 \left[e - \bar{\eta}_c(k) D\right] + \sum A_s'(f_s' - f_{cs})(e - \eta_s') - \sum A_s f_s (e + \eta_s) = 0 \qquad \text{for} \quad 0 < k < 1 \tag{27}$$

Compatibility equations:

$$\epsilon_s' = \frac{\epsilon_c}{kD}\left[\eta_s' - D(\tfrac{1}{2} - k)\right]$$

$$\epsilon_s = \frac{\epsilon_c}{kD}\left[\eta_s + D(\tfrac{1}{2} - k)\right] \tag{28}$$

Deflection:

$$\delta = \frac{\ell^2}{\pi^2}\frac{\epsilon_c}{kD} \tag{29}$$

In order to facilitate the numerical computations the dimensionless functions, $C_c(k)$, $\bar{\eta}_c(k)$ and $\epsilon_c$, of k have been plotted in Fig. 9(b).

In this paper, the behavior and ultimate loads of axially loaded columns and columns subjected to eccentric loads with small eccentricities are not discussed. The ultimate capacities of these columns are greatly affected by the variation of the strength of the concrete,[22] the strain rate and the plastic instability. These factors require further experimental verifications and theoretical studies.

For a series of eccentrically loaded columns tested by Hognested,[21] the writer has computed theoretical ultimate loads and deflections using the foregoing formulas. The average of the ratio's of actual to computed ultimate load for 95 square and cylindrical columns is 0.976, the standard deviation

22. "Effect of Size and Shape of Test Specimens on Compressive Strength of Concrete", by H. F. Gonnerman, Proceedings ASTM, Vol. 25, 1925, pp. 237-50.

being ±0.062. The actual ultimate loads were generally lower than the theoretical values and the actual deformations were generally higher. This, the writer belives, is partly due to the duration of the tests.

## Angle Changes and Deflections

In determining the angle changes and the deflections, the deformation caused by shear stress is neglected as being small compared to that produced by bending. The variation in position of the neutral axis (see Fig. 6) has also been neglected as being small compared to the raius of curvature at all points.

Under conditions of short-time static loads, the curvature $\chi$ is a definite function of the external bending moment M, which is in turn a function of the variable x giving the position of the section being considered. Thus

$$\frac{d^2y}{dx^2} = \chi\left[M(x)\right] \tag{30}$$

The integrations of this equation give the angle change and the deflection y as functions of x. Due to the complexity of non-linear characteristics of reinforced concrete members, it would in many cases be more practical to carry out the integrations semi-graphically.

## Collapse Loads of Reinforced Concrete Structures

It has been realized that the actual distribution of moments in a reinforced concrete structure at the limiting state is different from that determined by elastic theory. The plastic behavior of the structure allows a redistribution of moments which causes an increase in the load-carrying capacity of the structure. The increase varies from a small percentage to about thirty per cent more than the ultimate load obtained by conventional analysis. Test results on the destruction of continuous two-span beams reported by Kazinczy[23] showed that the increase is much greater in under-reinforced beams than in over-reinforced beams. These results were pointed out again by W. H. Glanville and F. G. Thomas[24], and A. L. L. Baker[25].

The test results of three single span under-reinforced beams, which were fixed at the ends and tested to destruction, are presented in the Appendix. The collapse loads of these beams were about thirty percent higher than the ultimate loads obtained by elastic theory.

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23. "Die Plastizitat des Eisenbetons", by F. B. Kazinczy, Beton und Eisen, Vol. 32, No. 5, March 1933, pp. 74-80.
24. "Studies in Reinforced Concrete. V.; Moment Redistribution in Reinforced Concrete", by W. H. Glanville and F. G. Thomas, Department of Scientific and Industrial Research, Building Research Tech. Paper No. 22, London, May 1939, 52 pp.
25. "A plastic Theory of Design for Ordinary Reinforced and Prestressed Concrete Including Moment-Redistribution in Continuous Members", by A. L. L. Baker, Mag. of Concrete Research, Vo. 1, No. 2, London, June 1949, pp. 57-66. and
"Recent Research in Reinforced Concrete, and Its Application to Design," by A. L. L. Baker, Journal of Institute of Civil Eng. Vol. 35-36, London, Feb. 1951, pp. 262-329.

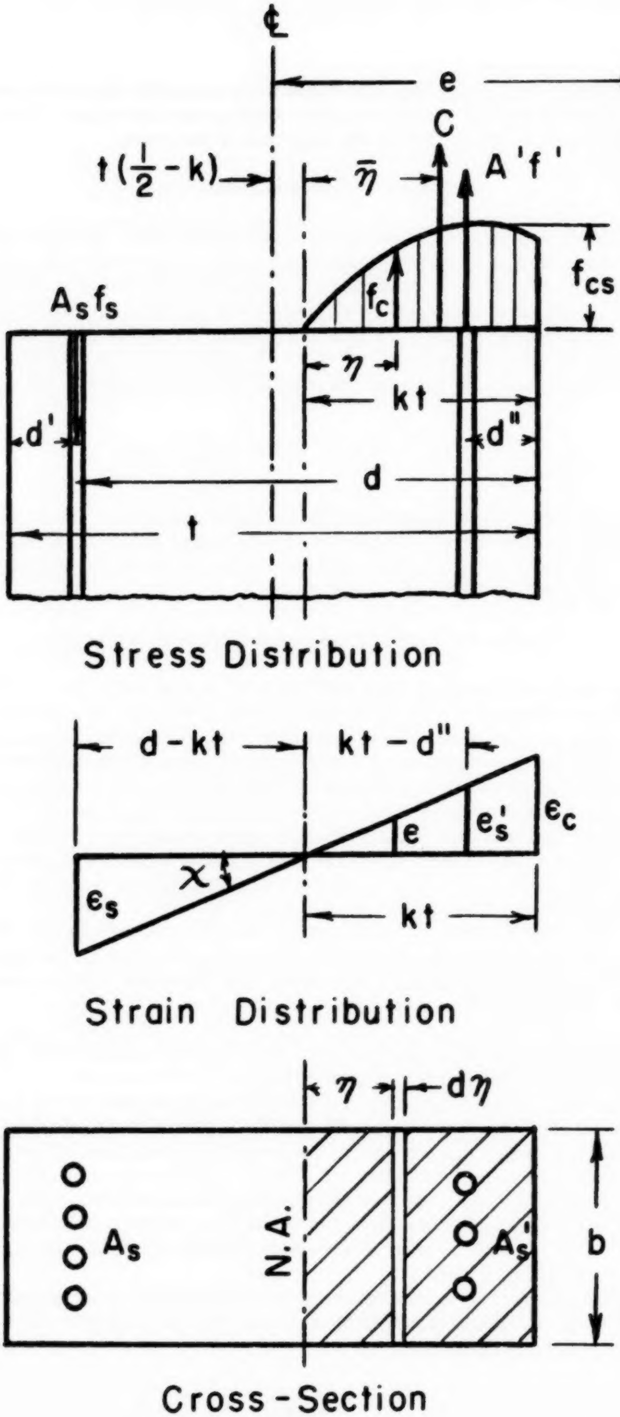


Fig. 8. Rectangular Section Subjected to Combined Axial Load and Bending.

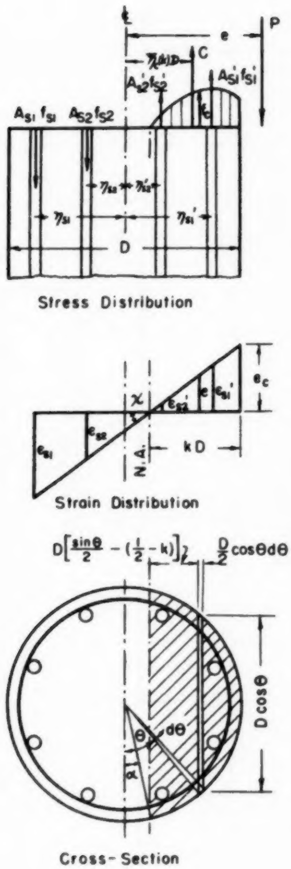


Fig. 9a. Circular Section Subjected to Combined Axial Load and Bending.

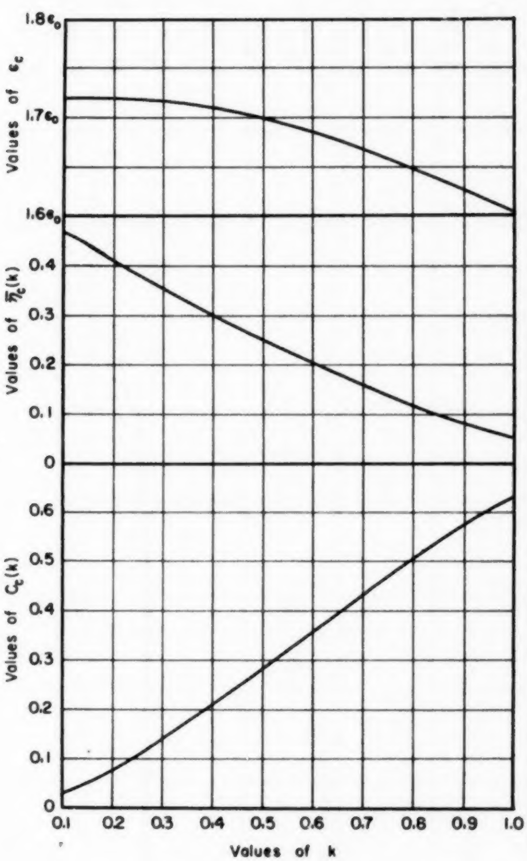


Fig. 9b. Functions of "k" for Analysis of Circular Sections.

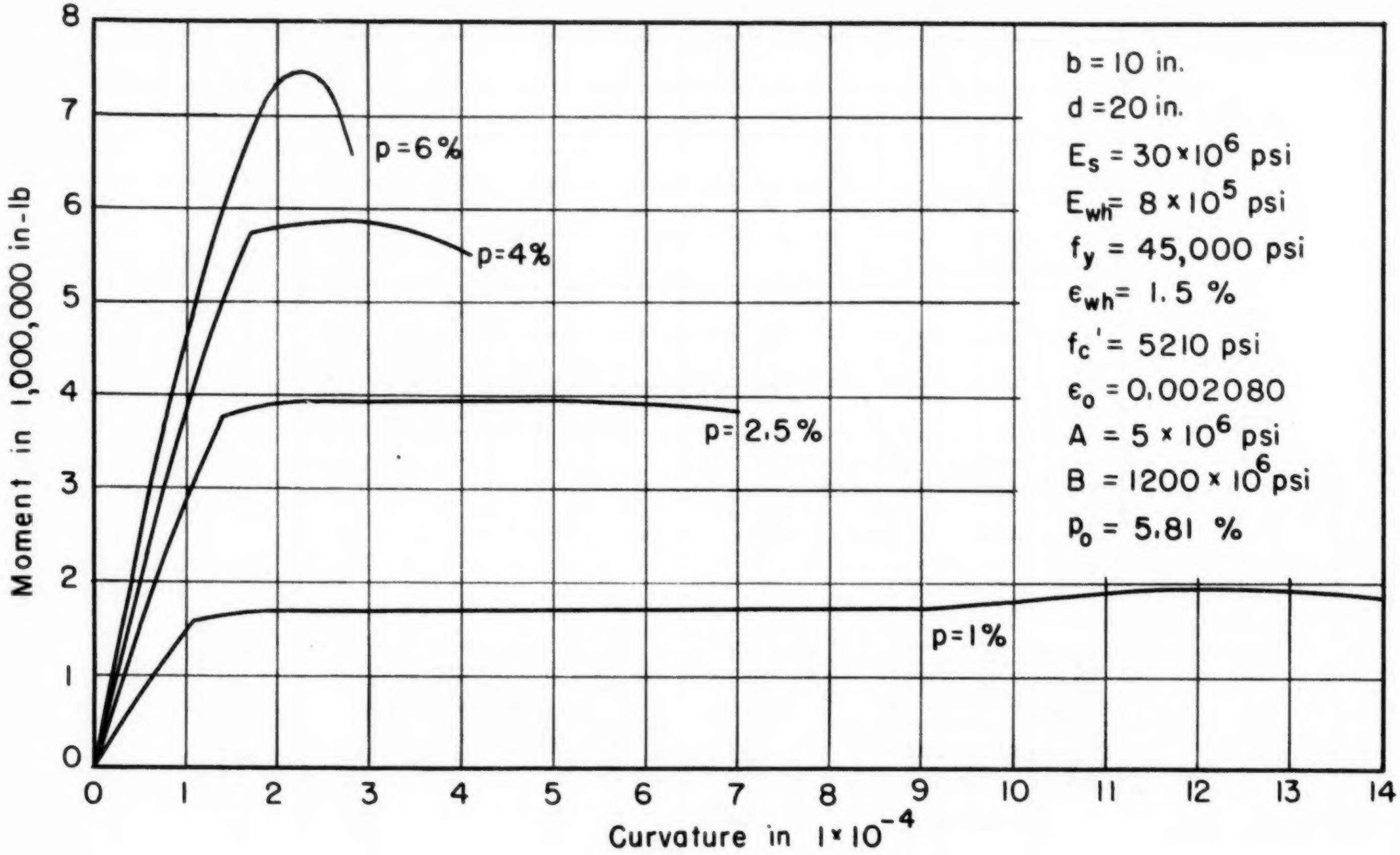


Fig. 10. Effect on Moment-Curvature Relation of Varying Steel Percentages in Rectangular Beams.

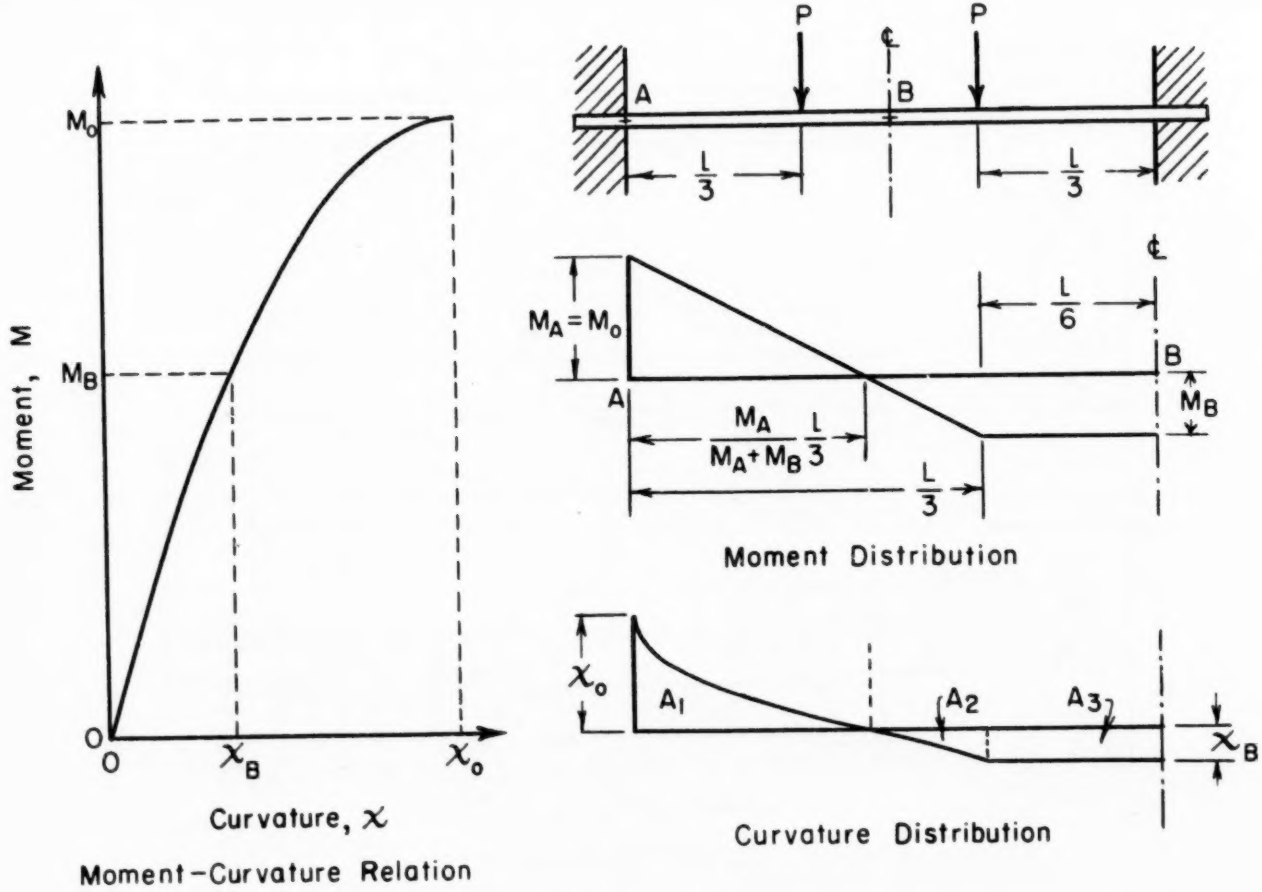


Fig. 11. Limiting State of a Beam Fixed at Both Ends.

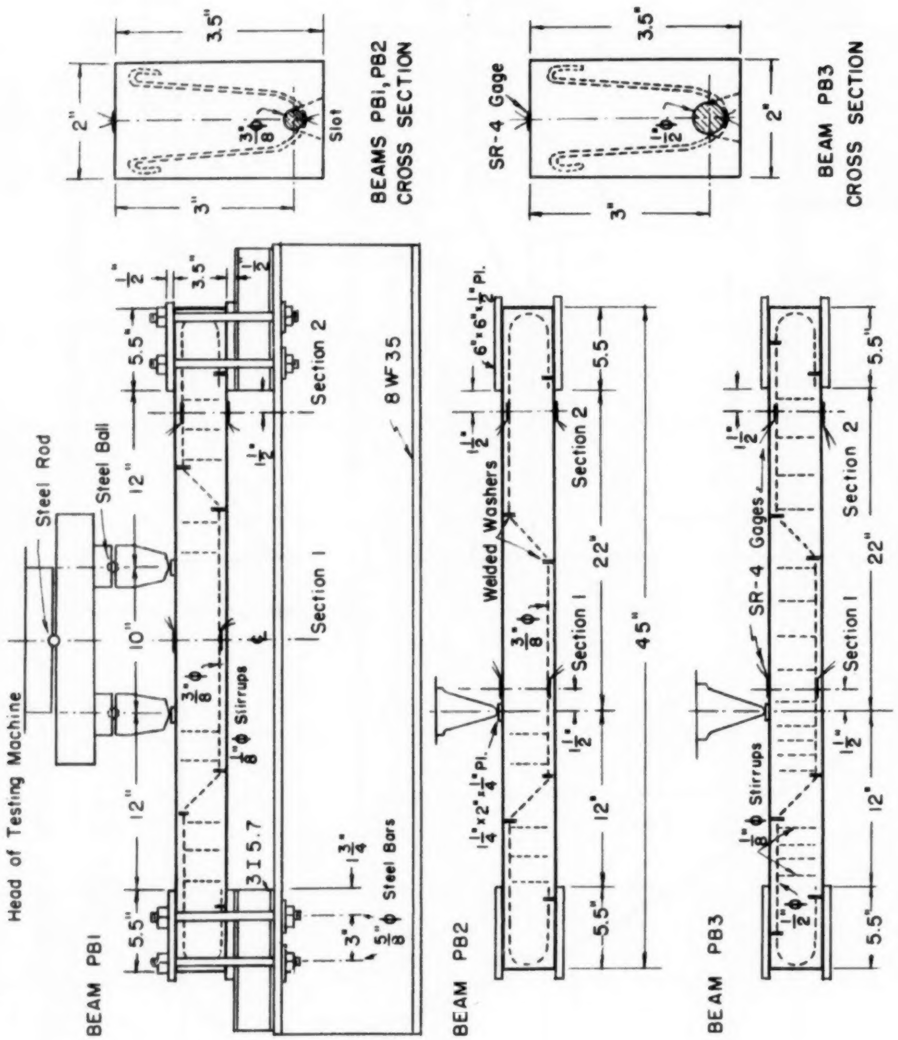


Fig. 12. Test Specimens.

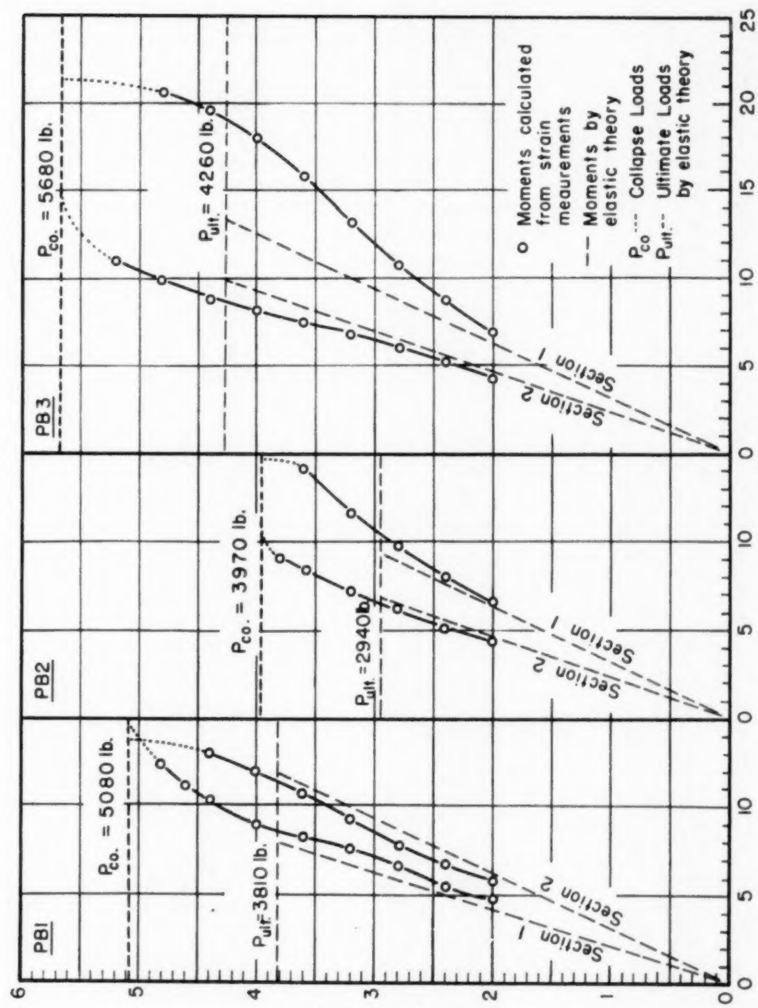


Fig. 13. Test Results.

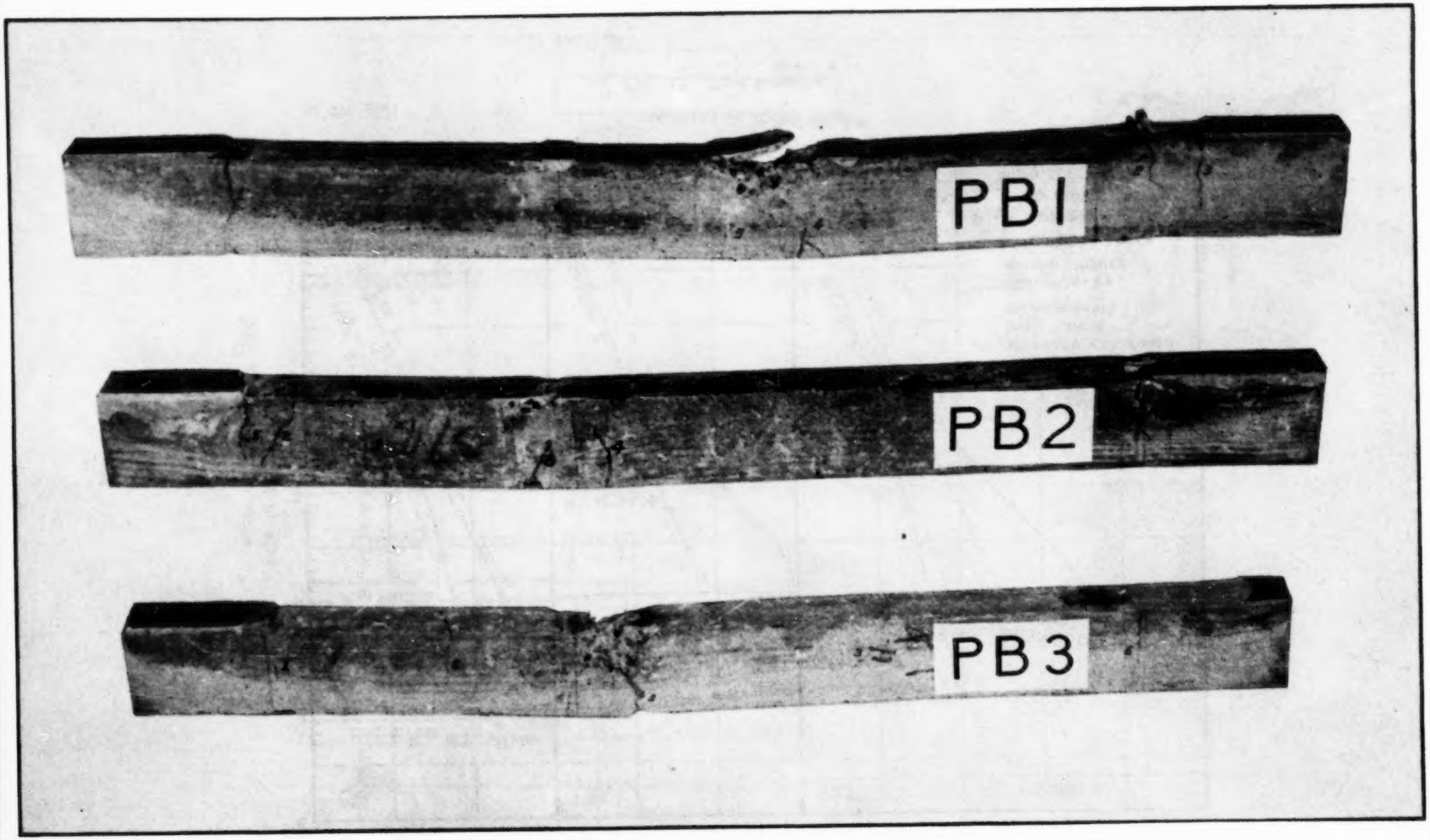


Fig. 14. Appearances of Beam Failures.

For more efficient use of the materials, it is necessary to determine collapse loads and corresponding distributions of moments for structures. In determining the limiting states, first the moment curvature relations of all members concerned are derived and next the expressions for the deflections and angle changes are developed. Once the moment-deflection relations and the moment-angle change relations are known, the determination of the distributions of moments may be carried out in the conventional manner, i.e., by solving simultaneous equations, involving angle changes and deflections, prescribed by the conditions of restraint.

The moment-curvature relation of a beam depends on many factors, but the characteristics of the relation are greatly influenced by the steel percentage. For example, the moment-curvature relations of four beams of the same size and the same concrete mix but with four different steel percentages were determined by Eqs. (8) and (9). The physical properties of the beams as well as the curves representing the relations are shown in Fig. 10. The curves terminate at the points where the extreme fiber strains reached the value of $2\epsilon_0$. Fig. 10 shows that the moment-curvature relation of the beam with low steel percentage resembles the yielding characteristics of steel, and the moment-curvature relation of the over-reinforced beam resembles the brittle characteristics of concrete.

The determination of the theoretical collapse load of a statically indeterminate beam is illustrated by the following example.

The moment-curvature relation for an over-reinforced beam, which is reinforced equally for both positive and negative bending, is assumed to be a parabola as shown in Fig. 11. The beam is absolutely fixed at both ends and subjected to loads at the third points. In order to obtain the collapse load, it is necessary to determine first the distribution of moments in the beam at incipient falure.

To satisfy the static conditions, the distribution of moments must be as shown in Fig. 11. The moment, $M_A$, at the fixed end can under no condition exceed the value of the maximum resisting moment, $M_O$. In the limiting state, the value of $M_A = M_O$. In this case, the value of $M_B$ is the only unknown and unknown and the distribution of curvature is shown in Fig. 11. The angle change of point B with respect to point A is zero, hence

$$\phi_{BA} = \int_A^B \chi\left[M(x)\right] \, dx = 0 \tag{31}$$

By solving Eq. (31), it is found that

$$M_B = 0.58\, M_o \tag{32}$$

and the corresponding collapse load is

$$P_{co} = \frac{4.74\, M_o}{\ell} \tag{33}$$

which shows that the collapse load for the over-reinforced beam subjected to the loads shown is 5.4 per cent higher than the ultimate load determined by the elastic theory. The result agrees closely with the findings by the previous investigators.

## Conclusions

On the basis of the studies reported herein, the following conclusions appear warranted:

1. Under short-time static loading conditions, the stress-strain relation of concrete in compression due to flexure is essentially similar to that obtained in direct compression. The stress-strain relation can be closely represented by a parabola.

2. The proposed inelastic theory gives satisfactory predictions of the ultimate loads of reinforced concrete beams and columns.

3. Within the scope of its assumptions, the theory provides a rigorous method of determining a complete picture of stress and deformation in statically indeterminate or determinate reinforced concrete members.

## Appendix

### Test Results

The tests were made with three beam specimens as shown in Fig. 12. The beams were designed to use the available testing equipment and to prevent any kind of failure except failure in flexure. The ends of the beams were fixed as shown in Fig. 12. The total time to test each beam was approximately 25 minutes.

The extreme fiber stresses with respect to the corresponding measured strains were computed by Eqs. (3) and (4). Computations for the stresses and moments of the specimen PB3 at Section 1 are illustrated in Table 1. The average stress-strain curve has been plotted in Fig. 2. The moment-curvature relations of the two sections were computed by Eqs. (8) and (9). In the computations, the values of $f_c'$ and $\epsilon_0$ from the average stress-strain curve were used. The moments at the sections corresponding to the measured curvatures were obtained from the theoretical moment-curvature relations. The variations of the moments at the six sections are shown in Fig. 13. For comparison, the moments computed by the elastic theory for all six sections are also shown in Fig. 13 in dashed lines. In computing the moments by elastic theory, the ends were assumed to be absolutely fixed and the external load was assumed to be concentrated at a point. Appearances of beam failures concentrated at a point. Appearances of beam failures are shown in Fig. 14.

### Acknowledgments

The tests described above were carried out in the Laboratories of the Civil Engineering and Engineering Mechanics Departments of the University of Notre Dame. The equipment, materials, and facilities were provided by the university. Thanks are also due to Dr. F. M. Kobayashi, Assistant Professor in Engineering Mechanics and Mr. Shun Hsu, Research Associate in Metallurgy, for their assistance in the experimental work, and to Dr. M. O. Peach, Associate Professor of Engineering Mechanics, for his critical reading of the paper.

Table 1. Test Results and Computations, Section 1, PB3

| Load | $\epsilon_c$ | $\epsilon_s$ | $\Sigma\epsilon_{(1)}$ | $f_{c(2)}$ | $\chi_{(3)}$ | M |
|---|---|---|---|---|---|---|
| lb | $1 \times 10^{-6}$ | $1 \times 10^{-6}$ | $1 \times 10^{-6}$ | psi | $1 \times 10^{-6}$ | in-lb |
| 0 | 0 | 0 | ——— | 0 | 0 | 0 |
| 2000 | 586 | 512 | ——— | ——— | 365 | 7,010 |
| 2200 | 658 | 569 | 1991 | 1875 | 407 | 7,800 |
| 2400 | 745 | 638 | 2240 | 2110 | 459 | 8,780 |
| 2600 | 833 | 706 | 2442 | 2300 | 510 | 9,700 |
| 2800 | 940 | 788 | 2716 | 2555 | 573 | 10,800 |
| 3000 | 1060 | 875 | 2911 | 2740 | 642 | 11,970 |
| 3200 | 1188 | 968 | 3236 | 3040 | 715 | 13,220 |
| 3400 | 1335 | 1067 | 3403 | 3210 | 798 | 14,550 |
| 3600 | 1481 | 1162 | 3640 | 3420 | 879 | 15,790 |
| 3800 | 1625 | 1250 | 3771 | 3540 | 945 | 16,800 |
| 4000 | 1779 | 1340 | 3945 | 3720 | 1038 | 18,080 |
| 4200 | 1921 | 1422 | 4174 | 3930 | 1110 | 18,920 |
| 4400 | 2042 | 1487 | 4183 | 3940 | 1170 | 19,650 |
| 4600 | 2144 | 1542 | 4365 | 4110 | 1227 | 20,350 |
| 4800 | 2230 | 1583 | 4156 | 3910 | 1265 | 20,650 |
| 5680 | Collapse Load | | | | | |

(1). $\Sigma\epsilon = \epsilon_c \frac{\Delta\epsilon_s}{\Delta\epsilon_c} + 2\epsilon_s \frac{\Delta\epsilon_s}{\Delta\epsilon_c} + \epsilon_s$, (2). $f_c = pE_s\Sigma\epsilon$, (3). $\chi = \frac{\epsilon_c + \epsilon_s}{3}$

$p = 0.0328$, $E_s = 28.7 \times 10^{-6}$ psi, $f_y = 46,400$ psi ($\frac{1}{2}$"$\phi$).